CHAPTER 5

TRIGONOMETRIC FUNCTIONS

5.1 Polar Coordinates and Trigonometry

PREREQUISITES

1. There are no prerequisites for this section other than basic high school geometry.

PREREQUISITE QUIZ

1. Use Quiz C on p. 14 to evaluate your preparation for this section. If you pass Quiz C, the material of this section may be reviewed quickly. If you have difficulties with Quiz C, you will need to study this section thoroughly.

GOALS

1. Be able to plot in polar coordinates.
2. Be able to write down the definitions of the trigonometric functions and to be able to find commonly used values.
3. Be able to make sketches of the graphs of the trigonometric functions.

STUDY HINTS

1. Radians. Calculus uses radians, not degrees! Be sure your calculator is in the right mode. Remember that 2π radians is 360° in order to make conversions.

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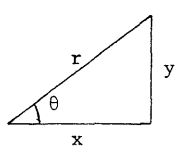
2. Equivalences. You should be able to convert back and forth from radians and degrees quickly in your head for the following angles: 0°, 30°, 45°, 60°, 90°, 180°, 270°, 360°. Use proportional reasoning: $90^\circ = 360^\circ/4$ is equivalent to $2\pi/4 = \pi/2$.

3. Negative polar coordinates. Remember that a negative angle represents clockwise motion and that a negative radius means reflection through the origin or a rotation of π radians, i.e., $(-r,\theta) = (r,\theta + \pi)$.

4. Definitions. Using the figure at the left, you should know the following definitions: $x/r = \cos\theta$, $y/r = \sin\theta$, $y/x = \tan\theta$, $\cot\theta = 1/\tan\theta$, $\sec\theta = 1/\cos\theta$, and $\csc\theta = 1/\sin\theta$. Note that if $r = 1$, $x = \cos\theta$ and $y = \sin\theta$. Also, no cofunction is the reciprocal of another cofunction. For example, $\csc\theta$ and $\cos\theta$ are not reciprocals. This may help you remember that $\sec\theta$ and $\cos\theta$ are reciprocals.



5. Commonly used values. Reproducing Fig. 5.1.17 and Fig. 5.1.18 will help you remember the most commonly used trigonometric values. If you have troubles remembering whether $\cos(\pi/6)$ is $1/2$ or $\sqrt{3}/2$, recall that $\cos\theta$ corresponds to x in the xy-plane. The cosine function decreases as θ increases on $[0,\pi/2]$. Thus, we expect $\cos(\pi/6) = \sqrt{3}/2$ and $\cos(\pi/3) = 1/2$. Similar analysis may be used for sine.

6. Calculuator errors. Inaccuracies may occur if θ is very large. For example, does $\cos(201\pi/2) = 0$ on your calculator?

7. Trigonometric identities. For the purposes of this course, you will find that $\cos^2\theta + \sin^2\theta = 1$ is very useful. Formulas (2), (3), and (4) will help you to derive other identities, but it is not an essential part of the course. Formulas (5) and (6) are especially important for integration

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7. Trigonometric identities (continued).

techniques which will be presented in Chapter 10. Other useful trigonometric identities may be found on the inside front cover of the text.

8. Period and frequency. A period is the length of a repeatable unit of a trigonometric graph. Frequency is the reciprocal of period.

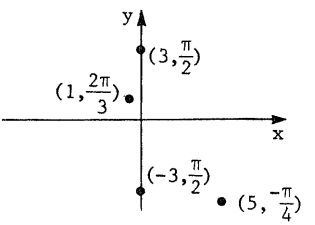
9. Graphing. Let TRIG be an arbitrary trigonometric function and let A and B be constants. The graph of $y = A\ TRIG\ Bx$ can be obtained by drawing $y = TRIG\ x$ and relabelling the x- and y-axes. All of the y-values should be multiplied by A and the x-values should be divided by B. The effect is stretching and compressing.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The problem is analogous to Example 1. We use the formulas $C_\theta = r\theta$ and $A_\theta = r^2\theta/2$. Remember that θ must be expressed in radians. We convert 22° to $22\times(\pi/180) = 0.3839$ radians. Thus, $C_\theta = r\theta = 10(0.3839) = 3.839$ m, and $A_\theta = (10)^2(0.3839)/2 = 19.195\ m^2$.

5. Since 2π radians $= 360^\circ$, the conversion factor is $2\pi/360 = \pi/180$. Thus, $29^\circ = 29(\pi/180) = 0.5061$ rad; $54^\circ = 54(\pi/180) = 0.9425$ rad; $255^\circ = 255(\pi/180) = 4.4506$ rad; $130^\circ = 130(\pi/180) = 2.2689$ rad; $320^\circ = 320(\pi/180) = 5.5851$ rad.

9.



In the notation (r,θ), r refers to the distance from the origin. If $r < 0$, rotate the point by π radians or reflect it across the origin. θ refers to the angle measured counterclockwise from the positive x-axis if $\theta > 0$; clockwise if $\theta < 0$.

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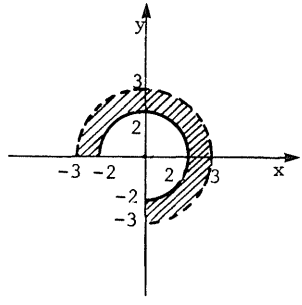
13. 

The region lies between circles of radii 2 and 3, centered at the origin. The circle of radius 2 is also included. In addition, the region is further restricted in that it must lie between angles of $-\pi/2$ and π, inclusive.

17. To make the conversion from cartesian coordinates to polar coordinates, use $r = \sqrt{x^2 + y^2}$, $\cos\theta = x/r$, and $\sin\theta = y/r$. θ can be found by using a trigonometric table or a calculator.

(a) $r = \sqrt{1^2 + 0^2} = 1$; $\cos\theta = 1$ implies $\theta = 0$, so the polar coordinates are $(1,0)$.

(b) $r = \sqrt{3^2 + 4^2} = 5$; $\cos\theta = 3/5$ implies $\theta = \pm 0.927$, but $\sin\theta = 4/5$ implies $\theta = 0.927$, so the polar coordinates are $(5,0.927)$.

(c) $r = \sqrt{3 + 1} = 2$; $\cos\theta = \sqrt{3}/2$ implies $\theta = \pm\pi/6$, but $\sin\theta = 1/2$ implies $\theta = \pi/6$, so the polar coordinates are $(2,\pi/6)$.

(d) $r = \sqrt{3 + (-1)^2} = 2$; $\cos\theta = \sqrt{3}/2$ implies $\theta = \pm\pi/6$, but $\sin\theta = -1/2$ implies $\theta = -\pi/6$, so the polar coordinates are $(2,-\pi/6)$.

(e) $r = \sqrt{(-\sqrt{3})^2 + (1)^2} = 2$; $\cos\theta = -\sqrt{3}/2$ implies $\theta = \pm 5\pi/6$, but $\sin\theta = 1/2$ implies $\theta = 5\pi/6$, so the polar coordinates are $(2,5\pi/6)$.

21. Use the formulas $x = r\cos\theta$ and $y = r\sin\theta$ to convert from polar coordinates to cartesian coordinates.

(a) $x = 6\cos(\pi/2) = 6(0) = 0$; $y = 6\sin(\pi/2) = 6(1) = 6$, so the cartesian coordinates are $(0,6)$.

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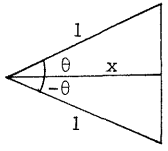
21. (b) $x = -12\cos(3\pi/4) = -12(-\sqrt{2}/2) = 6\sqrt{2}$; $y = -12\sin(3\pi/4) = -12(\sqrt{2}/2) = -6\sqrt{2}$, so the cartesian coordinates are $(6\sqrt{2}, -6\sqrt{2})$.

(c) $x = 4\cos(-\pi) = 4(-1) = -4$; $y = 4\sin(-\pi) = 4(0) = 0$, so the cartesian coordinates are $(-4,0)$.

(d) $x = 2\cos(13\pi/2) = 2\cos(\pi/2) = 2(0) = 0$; $y = 2\sin(13\pi/2) = 2\sin(\pi/2) = 2(1) = 2$, so the cartesian coordinates are $(0,2)$.

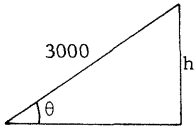
(e) $x = 8\cos(-2\pi/3) = 8(-1/2) = -4$; $y = 8\sin(-2\pi/3) = 8(-\sqrt{3}/2) = -4\sqrt{3}$, so the cartesian coordinates are $(-4, -4\sqrt{3})$.

(f) $x = -1\cos(2) = 0.42$; $y = -1\sin(2) = -0.91$, so the cartesian coordinates are $(0.42, -0.91)$.

25. 

Referring to the figure, $\cos\theta = x/1$ and $\cos(-\theta) = x/1$ also; therefore, $\cos\theta = \cos(-\theta)$.

29. By the definition of sine, we have $8/c = \sin(\pi/4) = \sqrt{2}/2$. Therefore, $c = 8/(\sqrt{2}/2) = 16/\sqrt{2} = 8\sqrt{2}$.

33. 

Let the height be h meters. Then $\tan 17^\circ = h/3000$ implies $h = 3000\tan 17^\circ \approx 917.19$ meters.

37. Use the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$ where θ is the measure of the angle (in radians) opposite c . Let x be the length of the unknown side. It is determined by $10^2 = x^2 + 5^2 - 2(x)(5)\cos(4\pi/5)$ or $0 = x^2 + 8.09x - 75$. Thus, the quadratic formula gives $x \approx 13.60$. Now, the law of cosines yields $5^2 = 10^2 + (13.60)^2 - 2(10)(13.60)\cos r$ or $\cos r = 0.956$, i.e., $r = 0.30$ radians.

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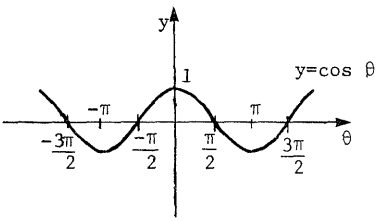
41. From the text, we get $\cos(\theta - \phi) = \cos\phi \cos\theta + \sin\phi \sin\theta$ and $\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$. Subtract to get $\cos(\theta - \phi) - \cos(\theta + \phi) = 2 \sin\theta \sin\phi$. Division by two gives the desired product formula.

45. By the addition formula, $\cos(3\pi/2 - \theta) = \cos(3\pi/2)\cos\theta + \sin(3\pi/2)\sin\theta = 0 \cdot \cos\theta + (-1) \sin\theta = -\sin\theta$.

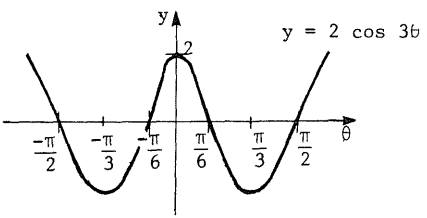
49. Using the addition formulas, we get $\cos(\theta + \pi/2)\sin(\phi - 3\pi/2) = [\cos\theta \cos(\pi/2) - \sin\theta \sin(\pi/2)] \cdot [\sin\phi \cos(3\pi/2) - \cos\phi \sin(3\pi/2)] = [0 \cdot \cos\theta - 1 \cdot \sin\theta] \cdot [0 \cdot \sin\phi - (-1) \cos\phi] = -\sin\theta \cos\phi$.

53. Use the half-angle formula. $\cos(\pi/12) = \cos[(\pi/6)/2] = \sqrt{[1 + \cos(\pi/6)]/2} = \sqrt{(1 + \sqrt{3}/2)/2} = \sqrt{2 + \sqrt{3}}/2$. Now $\sec(\pi/12) = 1/\cos(\pi/12) = 2/\sqrt{2 + \sqrt{3}}$.

57. Take the reciprocal of both sides and use the half-angle formula $\cos^2(\theta/2) = (1 + \cos\theta)/2$. Then $\sec^2(\theta/2) = (2 \sec\theta)/(\sec\theta + 1)$ implies $\cos^2(\theta/2) = (\sec\theta + 1)/(2 \sec\theta) = 1/2 + 1/(2 \sec\theta) = 1/2 + (\cos\theta)/2 = (1 + \cos\theta)/2 = \cos^2(\theta/2)$.

61. 

We obtain $y = 2 \cos 3\theta$ by compressing the graph of $y = \cos\theta$ horizontally by a factor of 3 and stretching it vertically by a factor of 2.



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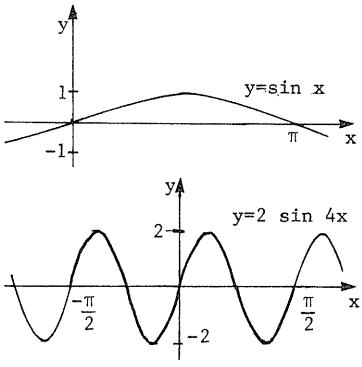
65.



We begin by simplifying $y = 4 \sin 2x \times \cos 2x = 2(2 \sin 2x \cos 2x) = 2(\sin 2(2x)) = 2 \sin 4x$. Hence, we obtain $y = 4 \sin 2x \cos 2x$ by compressing the graph of $y = \sin x$ horizontally by a factor of 4 and stretching it vertically by a factor of 2 .

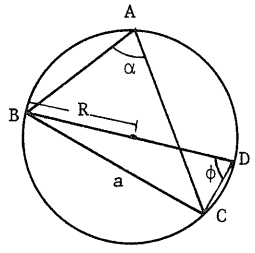
69. From Fig. 5.1.4(f), notice that the concavity changes at $-3\pi/2$, $-\pi/2$, $\pi/2$, and $3\pi/2$. In general, the inflection points for $\cot \theta$ are $(2n + 1)\pi/2$ where n is an integer.

73. (a) Rearrangement of Snell's law yields $v_1(\sin \theta_2/\sin \theta_1) = v_2 = (3 \times 10^{10})(\sin(30^\circ)/\sin(60^\circ)) = (3 \times 10^{10})[(1/2)/(\sqrt{3}/2)] = (3 \times 10^{10})/\sqrt{3} = \sqrt{3} \times 10^{10}$ cm/sec .

(b) If $v_1 = v_2$, then $\sin \theta_1 = \sin \theta_2$ implies $\theta_1 = \theta_2$. Therefore the angle of incidence is the same as the angle of refraction, so the light travels in a straight line.

(c) We have $v_2 = v_1/2$, so $v_1/v_2 = 2 = \sin(45^\circ)/\sin \theta_2 = (\sqrt{2}/2)/\sin \theta_2$. This implies $\sin \theta_2 = \sqrt{2}/4$, so $\theta_2 = 0.36$ radians or 20.7° .

77.



We will show that $\sin \alpha / a = 1/2R$ where R is the radius of the circumscribed circle. Let $\overline{BD}$ be a diameter of the circle, then geometry tells us that $\angle BCD$ is a right angle. Therefore, $\sin \phi = a/2R$. Angle α and angle ϕ subtend the same arc, so $\alpha = \phi$ and $\sin \alpha = a/2R$. Consequently, $\sin \alpha/a = 1/2R$.

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81. (a) When $\sin\theta = \lambda/a$, we have $I = I_0\{\sin[\pi a(\lambda/a)/\lambda]/[\pi a(\lambda/a)/\lambda]\}^2 = I_0(\sin\pi/\pi)^2 = I_0(0/\pi)^2 = 0$.

(b) $\sin\phi = 0$ for $\phi = n\pi$, where n is an integer; therefore, we solve $\pi a \sin\theta/\lambda = n\pi$. The solution is $\sin\theta = n\lambda/a$, so θ is all positive values whose sine is $n\lambda/a$.

(c) When $\pi a \sin\theta/\lambda$ is close to zero, Exercise 74 tells us that the numerator of the squared term is approximately equal to its denominator, so the squared term is approximately 1. Therefore, I is approximately I_0.

(d) $\sin\theta = (5 \times 10^{-5})/(10^{-2}) = 5 \times 10^{-3}$ implies $\theta = 5.000021 \times 10^{-3} \approx 5 \times 10^{-3} = \lambda/a$.

SECTION QUIZ

1. $(-4,4)$ is a point described in cartesian coordinates. What are its corresponding polar coordinates?

2. Suppose $(r,\theta) = (\pi/4,1)$ are the polar coordinates of a point.

   (a) Use a negative radius to describe the same point.

   (b) Convert to cartesian coordinates, rounding to the nearest 0.1.

3. Let $(3,-4)$ be the cartesian coordinates of a point and let (r,θ) be the polar coordinates. Without using a calculator or tables, find:

   (a) $\sin\theta$

   (b) $\sec\theta$

4. Sketch $y = \cos x$. Trace the graph and relabel the axes to depict $y = (1/3)\cos 2x$.

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5. To celebrate the end of another midterm, Merry May decided to throw a party. Since polar coordinates were still on her mind, she located the beginning of a song on a record at the polar coordinates $A(5,-\pi/4)$; the song ends at the polar coordinates $C(3,3\pi/4)$. Another song begins at point B which is -3 units from A at an angle of $-\pi/6$.

   (a) What are the cartesian coordinates of A, B, and C?

   (b) Suppose Merry May only had a straightedge to help her locate songs. Plot all three points and help her determine the distance from C to B.

ANSWERS TO SECTION QUIZ

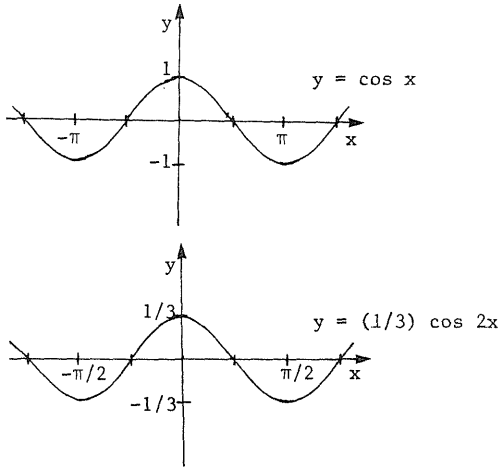
1. $(4\sqrt{2},3\pi/4)$

2. (a) $(-\pi/4,1+\pi)$

   (b) $(0.4,0.7)$

3. (a) $-4/5$

   (b) $5/3$

4. 

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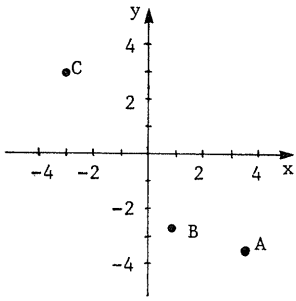
5. (a) $A(5\sqrt{2}/2, -5\sqrt{2}/2)$

$B((5\sqrt{2} - 3\sqrt{3})/2, (-5\sqrt{2} + 3)/2)$

$C(-3\sqrt{2}/2, 3\sqrt{2}/2)$

(b) $[73 - 12\sqrt{6} - 12\sqrt{2}]^{1/2}$



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5.2 Differentiation of the Trigonometric Functions

PREREQUISITES

1. Recall how to define the trigonometric functions in terms of sines and cosines (Section 5.1).
2. Recall the definition of the derivative (Section 1.3).
3. Recall the basic rules of differentiation (Sections 1.4, 1.5, and 2.2).
4. Recall how to compute an antiderivative and how to use the fundamental theorem of calculus (Sections 2.5 and 4.4).

PREREQUISITE QUIZ

1. Express the following in terms of $\sin x$ and $\cos x$:
   (a) $\tan x$
   (b) $\cot x$
   (c) $\sec x$
   (d) $\csc x$
2. What is the relationship between $f(x_0)$, Δx , and $f'(x_0)$?
3. Differentiate the following functions:
   (a) $(x^2 + 2x + 2)(x^2 + 3x)$
   (b) $(x^2 + 2x + 2)/(x^2 + 3x)$
   (c) $(x^3 + 3x - 5)^5$
4. State the fundamental theorem of calculus.
5. Suppose $(d/dx)\Delta(x) = \delta(x)$; what is $\int\delta(x)dx$?

GOALS

1. Be able to differentiate trigonometric functions.
2. Be able to integrate certain trigonometric functions.

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STUDY HINTS

1. Derivative of sine, cosine. For the purposes of this course, you should memorize that the derivative of $\sin x$ and $\cos x$ are $\cos x$ and $-\sin x$, respectively. It is not essential to know the derivation. Don't forget the minus sign in $(d/dx)\cos x = -\sin x$.

2. Other derivatives of trigonometric functions. You may find it easier to write the other trigonometric functions in terms of $\sin$ and $\cos$ and then use the rules of differentiation to derive the derivatives. On the other hand, you may prefer to memorize the differentiation formulas. One can simply memorize the derivatives of $\tan$ and $\sec$ and then remember that the derivatives of the cofunctions are the negative cofunctions. For example, we know that $(d/dx)\tan x = \sec^2 x$. Then, since $\cot x$ and $\csc x$ are the cofunctions of $\tan x$ and $\sec x$, respectively, we get $(d/dx)\cot x = -\csc^2 x$.

3. Trigonometric antiderivatives. There is really nothing new to learn, but it is important to get used to the notation and to reading the differentiation formulas backwards. The antidifferentiation formulas follow directly from the differentiation formulas derived in this section.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. By the sum rule, $(d/d\theta)(\cos\theta + \sin\theta) = -\sin\theta + \cos\theta$.

5. By the product rule, $(d/d\theta)[(\cos\theta)(\sin\theta + \theta)] = (-\sin\theta)(\sin\theta + \theta) + (\cos\theta)(\cos\theta + 1) = \cos^2\theta - \sin^2\theta - \theta\sin\theta + \cos\theta = \cos 2\theta - \theta\sin\theta + \cos\theta$.

9. By the quotient rule, $(d/d\theta)[\cos\theta/(\cos\theta - 1)] = [(-\sin\theta)(\cos\theta - 1) - \cos\theta(-\sin\theta)]/(\cos\theta - 1)^2 = \sin\theta/(\cos\theta - 1)^2$.

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13. By the chain rule, $(d/dx)(\cos x)^3 = 3(\cos x)^2(-\sin x) = -3\cos^2 x \sin x$.

17. By the chain rule, $(d/dx)\sin(x + \sqrt{x}) = [\cos(x + \sqrt{x})](1 + 1/2\sqrt{x})$.

21. By the sum rule, $(d/dx)(\tan x + 2\cos x) = \sec^2 x - 2\sin x$.

25. $df(x)/dx = 1/2\sqrt{x} - 3\sin 3x$ by applying the chain rule to $\cos 3x$.

29. $df(t)/dt = (12t^2)\sin\sqrt{t} + (4t^3 + 1)(\cos\sqrt{t})(1/2\sqrt{t}) = 12t^2\sin\sqrt{t} + [(4t^3 + 1)/2\sqrt{t}]\cos\sqrt{t}$. The chain rule and the product rule were used.

33. Apply the chain rule and quotient rule to get $df(\theta)/d\theta = (3/2)[\csc(\theta/\sqrt{\theta^2 + 1}) + 1]^{1/2}[-\cot(\theta/\sqrt{\theta^2 + 1})\csc(\theta/\sqrt{\theta^2 + 1})]\cdot\{[\sqrt{\theta^2 + 1} - \theta(1/2)(\theta^2 + 1)^{-1/2}(2\theta)]/(\theta^2 + 1)\} = -3[\csc(\theta/\sqrt{\theta^2 + 1}) + 1]^{1/2}[\cot(\theta/\sqrt{\theta^2 + 1})][\csc(\theta/\sqrt{\theta^2 + 1})]/2(\theta^2 + 1)^{3/2}$.

37. $\int(x^3 + \sin x)dx = x^4/4 - \cos x + C$.

41. Guess that the antiderivative is $a\cos(u/2)$. Differentiation gives $-(1/2)a\sin(u/2)$, so $a = -2$; therefore $\int\sin(u/2)du = -2\cos(u/2) + C$.

45. The useful trigonometric identity is $\sin 2\theta = 2\sin\theta\cos\theta$, so $\int\cos\theta\sin\theta\, d\theta = (1/2)\int\sin 2\theta d\theta$. Guess that the antiderivative has the form $a\cos 2\theta$. Differentiation gives $-2a\cos 2\theta$, so $a = -1/2$. Therefore, the integral is $(1/2)(-1/2)\cos 2\theta + C = -\cos 2\theta/4 + C$.

49. The antiderivative should have the form $a\cos(\theta/4)$. Differentiation gives $-(a/4)\sin(\theta/4)$, so $a = -4$; therefore, $\int_0^{\pi/2}\sin(\theta/4)d\theta = -4\cos(\theta/4)\Big|_0^{\pi/2} = 4 - 4\cos(\pi/8)$.

53. Differentiating $a\sin(3\pi t)$ gives $3\pi a\cos(3\pi t)$. This implies that $\int_0^1\cos(3\pi t)dt = \sin(3\pi t)/3\pi\Big|_0^1 = (\sin 3\pi - \sin 0)/3\pi = 0$.

57. (a) Note that $\sin 2\phi/(1 + \cos 2\phi) = 2\sin\phi\cos\phi/(1 + 2\cos^2\phi - 1) = \sin\phi/\cos\phi = (\sin\phi/\phi)\cdot(\phi/\cos\phi)$. Since $\cos\phi < \sin\phi/\phi < 1$ for $0 < \phi < \pi/2$, we have $\cos\phi\cdot(\phi/\cos\phi) < (\sin\phi/\phi)\cdot(\phi/\cos\phi)$. That is, $\phi < (\sin\phi/\phi)\cdot(\phi/\cos\phi) = \sin 2\phi/(1 + \cos 2\phi)$ for $0 < \phi < \pi/2$.

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57. (b) Note that $\sec \phi = 1/\cos \phi$. Since $\cos \phi < \sin \phi/\phi$ and $\sin \phi < 1$ for $0 < \phi < \pi/2$, $\cos \phi < 1/\phi$, i.e., $1/\cos \phi > \phi$. Hence $\sec \phi = 1/\cos \phi > \phi$.

61. Let $f(x) = \sin x$, then $f'(x) = \cos x$ and $f''(x) = -\sin x$, so $f''(x) + f(x) = 0$. Let $f(x) = \cos x$, then $f'(x) = -\sin x$ and $f''(x) = -\cos x$, so $f''(x) + f(x) = 0$.

65. Differentiate by using the chain rule. $(d/d\theta)[-f(\cos \theta)] = -f'(\cos \theta) \times [(d/d\theta)\cos \theta]$. Since $f'(x) = 1/x$, we have $-(1/\cos \theta)(-\sin \theta) = \tan \theta$. Thus, $-f(\cos \theta) + C$ is the antiderivative for $\tan \theta$.

69. (a) Using the product rule, $(d/dx)[\phi(3x)\cos x] = \phi(3x)\cdot(-\sin x) + (\cos x)\phi'(3x)3 = -(\sin x)\cdot\phi(3x) + 3\cos x/\cos(3x)$.

(b) Since $d\phi/dx = 1/\cos x$, an antiderivative for $1/\cos x$ is $\phi(x)$. Thus, $\int_0^1 (dx/\cos x) = \phi(x)\Big|_0^1 = \phi(1) - \phi(0)$.

(c) Using the product rule, $(d/dx)(\phi(2x)\sin 2x) = 2\phi'(2x)\cdot\sin 2x + \phi(2x) \times (2\cos 2x) = 2\sin 2x/\cos 2x + 2\phi(2x)\cos 2x$. Differentiating again, $(d^2/dx^2)(\phi(2x)\sin 2x) = [4\cos 2x(\cos 2x) - 2\sin 2x(-2\sin 2x)]/\cos^2 2x + 2\cdot 2\phi'(2x)\cos 2x + 2\phi(2x)\cdot(-2\sin 2x) = 4(\cos^2 2x + \sin^2 2x)/\cos^2 2x + 4\cos 2x/\cos 2x - 4\phi(2x)\cdot\sin 2x = 4\sec^2 2x + 4 - 4\phi(2x)\cdot\sin 2x$.

SECTION QUIZ

1. What is wrong with this statement? If $f(x) = \sin^3 3x$, then $f'(x) = 3 \sin^2 3x \cos 3x$.

2. Differentiate the following with respect to x:

(a) $\cos x \sin x$

(b) $\sec(\pi/4)$

(c) $\tan(x^3)$

(d) $\sec^2 x$

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3. Evaluate the following integrals:

   (a) $\int_0^{\pi} \sin 3x \, dx$

   (b) $\int_0^{\pi/2} \cos \pi \, dx$

4. As usual, the Do-Wrong Construction Company did wrong again. Just yesterday, it finished building a storage room for nuclear reactor parts; however, Do-Wrong forgot to make a door. Due to expense, it is desirable to remove as little wall material as possible. The largest nuclear reactor part has the shape of the region under $y = \tan^2 x$ for $\pi/6 \leq x \leq \pi/3$. How much of the wall needs to be removed? (Hint: $\tan^2\theta + 1 = \sec^2\theta$.)

ANSWERS TO PREREQUISITE QUIZ

1. (a) $\sin x/\cos x$

   (b) $\cos x/\sin x$

   (c) $1/\cos x$

   (d) $1/\sin x$

2. $f'(x_0) = \lim_{\Delta x \to 0} \{[f(x_0 + \Delta x) - f(x_0)]/\Delta x\}$

3. (a) $(2x + 2)(x^2 + 3x) + (x^2 + 2x + 2)(2x + 3)$

   (b) $[(2x + 2)(x^2 + 3x) - (x^2 + 2x + 2)(2x + 3)]/(x^2 + 3x)^2$

   (c) $15(x^3 + 3x - 5)^4(x^2 + 1)$

4. If $F'(x) = f(x)$, then $\int_a^b f(x)dx = F(x)\Big|_a^b$.

5. $\Delta(x) + C$

ANSWERS TO SECTION QUIZ

1. $f'(x) = 9 \sin^2 3x \cos 3x$; 9 comes from the exponent 3 and the derivative of $3x$.

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2. (a) $\cos^2 x - \sin^2 x$

   (b) 0

   (c) $3x^2 \sec^2(x^3)$

   (d) $2 \sec^2 x \tan x$

3. (a) $2/3$

   (b) $-\pi/2$

4. $2\sqrt{3}/3 - \pi/6$

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5.3 Inverse Functions

PREREQUISITES

1. Recall how to use the vertical line test (Section R.6).
2. Recall how to use the intermediate value theorem (Section 3.1).
3. Recall how to evaluate composite functions and how to differentiate them (Section 2.2).
4. Recall the definition of the derivative (Section 1.3).

PREREQUISITE QUIZ

1. Suppose $f(x) = x^8$, what is $\lim_{\Delta x \to 0} \{ [f(x + \Delta x) - f(x)] / \Delta x \}$?
2. If $f(0) = 1$ and $f(2) = -2$, and f is continuous, can you locate a root of f ? Explain.
3. Let $f(x) = x^2$ and $g(x) = \cos x$.
   (a) Find $f(g(x))$ and its derivative.
   (b) Find $g \circ f(x)$ and its derivative.
4. Explain the vertical line test.

GOALS

1. Be able to state and explain the definition of an inverse.
2. Be able to differentiate simple inverse functions.

STUDY HINTS

1. Inverses. For most cases, we express a function by $y = f(x)$. If it is possible to uniquely find x in terms of y , then we say that x is the inverse of y . However, in many instances, an inverse may exist even if we cannot explicitly find x in terms of y . The graphs .

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1. Inverses (continued).

of inverses may be obtained by "flipping" the axes. See Fig. 5.3.1.

2. Horizontal line test. If each horizontal line meets the graph of f in at most one point, then f is invertible. Compare how this statement corresponds to the vertical line test (See Section R.6). How does "flipping" axes make this statement plausible?

3. Inverse function test. All the test says is that any part of a function that is strictly increasing or strictly decreasing is invertible. Think about what problem exists if the function is not strictly increasing or strictly decreasing. The domain is $[f(a),f(b)]$ or $[f(b),f(a)]$; just be sure the smaller number comes first.

4. Inverse function rule. Probably the easiest formula to remember is $dx/dy = 1/(dy/dx)$ because the differentials act like regular fractions. The formula is very easy to use if you are differentiating at a specific point (See Example 8). If you want to differentiate an inverse function, you need to know both the function and its inverse. The best way to learn to use the formula is to study Example 7 and practice. (You will get more practice in the next section.)

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. $f'(x) = 2$ does not change sign, so $f(x)$ has an inverse. We have $f(x) = y = 2x + 5$, $f(-4) = -3$, and $f(4) = 13$. Solving for the independent variable in terms of the dependent variable, we get $x = (y - 5)/2$. Changing the names of the variables yields $f^{-1}(x) = (x - 5)/2$ on $[-3,13]$.

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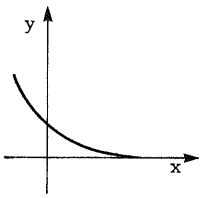
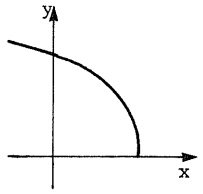
5. $h'(t) = [(t + 3) - (t - 10)]/(t + 3)^2 = 13/(t + 3)^2$ does not change sign, so $h(t)$ has an inverse. We have $h(t) = (t - 10)/(t + 3)$, $h(-1) = -11/2$, and $h(1) = -9/4$. Solving for the independent variable in terms of the dependent variable, we get $ht + 3h = t - 10$ or $t(h - 1) = -3h - 10$. Thus, the inverse is $t = (3h + 10)/(1 - h)$ or $h^{-1}(t) = (3t + 10)/(1 - t)$ on $[-11/2,-9/4]$.

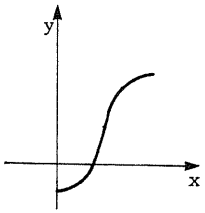
9. (a)



(b)

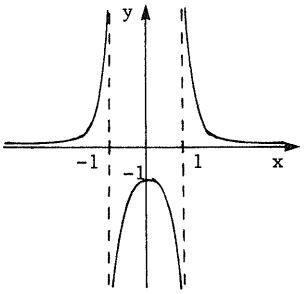


(c)

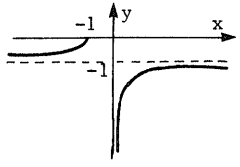


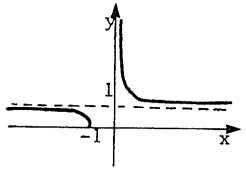
The inverses were graphed by tracings through the back of the page and then rotating the x-axis into a vertical position.

13.



$f'(x) = -(x^2 - 1)^{-2} \cdot 2x$, so the critical point is $x = 0$. Thus, f is increasing on $(-\infty,-1)$ and $(-1,0)$. It is decreasing on $(0,1)$ and $(1,\infty)$. From the graph at the left, we see that the largest interval on which f is invertible is $(-\infty,-1)$ or $(1,\infty)$. The graphs of the inverses are shown below and were obtained by flipping through the line $x = y$.





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17. We apply the inverse function test. $f'(x) = x^2 - 1$, which is zero if $x = \pm 1$. f is increasing on $(-\infty,-1)$ and $(1,\infty)$, and decreasing on $(-1,1)$. Hence f is not strictly increasing or decreasing on any open interval containing 1. That is, f is not invertible on such an interval.

21. $f'(x) = -3x^2 - 2$, which is negative for all x. Thus, f is decreasing on $[-1,2]$. Also, f is continuous, so by the inverse function test, f is invertible on $[-1,2]$. The domain of the inverse is $[f(2),f(-1)]$, i.e., $[-11,4]$. The domain is $[f(b),f(a)]$ since f is decreasing.

25. Since f is a polynomial, it is continuous. The derivative is $f'(x) = 3x^2 + 2$, which is always positive. By the inverse function test, f has an inverse $g(y)$. Notice that $g(4) = 1$ since $f(1) = 4$. From the inverse function rule, with $y_0 = 4$ and $x_0 = 1$, $g'(4) = 1/f'(1) = 1/[3(1)^2 + 2] = 1/5$.

29. $f'(x) = 3$ for all x, so by the inverse function rule, $g'(y) = 1/f'(x) = 1/3$ for all x. Thus, we have $g'(2) = g'(3/4) = 1/3$.

33. The errors do not seem to build up on most calculators. In most cases, you should get back $|x|$, which is not the original number if you started with a negative number. This demonstration does not work if x is too small or too large. Notice that $\sqrt{x}$ and x^2 are inverses only if $x \geqslant 0$.

37. (a) Suppose $3x + 7 > 0$ or $x > -7/3$, then $2x + 5 \geqslant 0$ or $x \geqslant -5/2$; therefore, we must have $x > -7/3$. Now, if $3x + 7 < 0$ or $x < -7/3$, then $2x + 5 \leqslant 0$ or $x \leqslant -5/2$; therefore, we must have $x \leqslant -5/2$. Thus, the domain is $(-\infty,-5/2]$ and $(-7/3,+\infty)$.

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37. (b) An equivalent expression is $y^2 = (2x + 5)/(3x + 7)$, which implies $3xy^2 + 7y^2 = 2x + 5$ or $x(3y^2 - 2) = 5 - 7y^2$. Therefore, the inverse function is $g(y) = (5 - 7y^2)/(3y^2 - 2)$.

(c) $f'(x) = 1/g'(y) = \{[(-14y)(3y^2 - 2) - (5 - 7y^2)(6y)]/(3y^2 - 2)^2\}^{-1} = -(3y^2 - 2)^2/2y = -[3((2x + 5)/(3x + 7)) - 2]^2/2\sqrt{(2x + 5)(3x + 7)} = -[1/(3x + 7)^2]/2\sqrt{(2x + 5)(3x + 7)} = -1/[2(3x + 7)^{3/2}\sqrt{2x + 5}]$.

(d) $f'(x) = (1/2)[(2x + 5)/(3x + 7)]^{-1/2}[(2)(3x + 7) - (2x + 5)(3)]/(3x + 7)^2 = -1/[2(3x + 7)^{3/2}\sqrt{2x + 5}]$.

41. (a) By the definition of the derivative, $f'(x_0) = \lim_{\Delta x \to 0}(\Delta y/\Delta x)$. Thus, as Δx gets smaller, $\Delta y/\Delta x$ approaches $f'(x_0)$, which is positive according to the hypothesis. Therefore, we have $(3/2)f'(x_0) > \Delta y/\Delta x > (1/2)f'(x_0)$ for Δx sufficiently small.

(b) Let us assume Δx does not approach 0 ; then $\Delta y/\Delta x$ must approach 0 as Δy approaches 0 . This cannot occur because $\Delta y/\Delta x$ is between two positive numbers.

(c) $\Delta x = x - x_0 = g(y_0 + \Delta y) - g(y_0)$. Then $g(y_0 + \Delta y) = g(y_0) + \Delta x$ approaches $g(y_0)$ as Δy approaches zero since $\Delta x \to 0$. Thus g is continuous at y_0 .

SECTION QUIZ

1. Suppose $f(x) = (x^3 + 2x + 8)/x$. On what intervals is f invertible?
2. For the function in Question 1, one of the intervals contains $x = 1$. What is the derivative of f^{-1} at that point?
3. Find a formula for f^{-1} if $f(x) = (x - 1)/(x + 5)$.

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4. Backwards Billy was born with a neurological disease which caused him to think backwards. When asked to sketch the graph of $y = x^4 - 3x^3 + x^2 + 8$, he wants to start by sketching the inverse and then flipping the graph. However, you realize that y doesn't have an inverse.
   (a) Explain to Backwards Billy why the inverse doesn't exist.
   (b) Find the largest interval around $x = 1$ for which an inverse exists.
   (c) On the restricted interval, what is the derivative of the inverse at $x = 1$?

ANSWERS TO PREREQUISITE QUIZ

1. $8x^7$
2. By the intermediate value theorem, there is at least one root in $(0,2)$.
3. $f(g(x)) = \cos^2 x$ and $f'(g(x)) = -2 \sin x \cos x$; $g \circ f(x) = \cos(x^2)$ and $g'(f(x)) = -2x \sin(x^2)$.
4. If a vertical line drawn at each value of x intersects a graph at only one point, then it is the graph of a function.

ANSWERS TO SECTION QUIZ

1. $(-\infty,0)$, $(0,\sqrt[3]{4})$, and $(\sqrt[3]{4},\infty)$
2. $-1/6$
3. $f^{-1}(x) = (5x + 1)/(1 - x)$
4. (a) The inverse function test is violated; $f'(x) > 0$ on $(0,1/4)$ and $(2,\infty)$ and $f'(x) < 0$ on $(-\infty,0)$ and $(1/4,2)$.
   (b) $[1/4,2]$
   (c) $-1/3$

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5.4 The Inverse Trigonometric Functions

PREREQUISITES

1. Recall the concept of an inverse function (Section 5.3).
2. Recall how to differentiate an inverse function (Section 5.3).
3. Recall how to differentiate trigonometric functions (Section 5.2).

PREREQUISITE QUIZ

1. Evaluate the following:
   (a) $(d/dx) \sin x$
   (b) $(d/dy) \tan y$
   (c) $(d/dx)f^{-1}(x)\big|_2$ if $f(x) = x^3 - x$
2. On what intervals does $f(x) = x^4 + x^2$ have an inverse?
3. If $g(u) = u^3 + u$, what is $g^{-1}(2)$?

GOALS

1. Be able to differentiate the inverse trigonometric functions.
2. Be able to simplify trigonometric functions of inverses, i.e., what is $\tan(\sin^{-1}x)$?
3. Be able to integrate certain algebraic expressions that lead to inverse trigonometric functions.

STUDY HINTS

1. Notation. In this text and most others, $\sin^{-1}x$ is the inverse function of $\sin x$, not $1/\sin x$. Do not confuse this notation with $\sin^2 x$ or $\sin^3 x$, which mean $(\sin x)^2$ and $(\sin x)^3$, respectively. The notation arcsin, arccos, etc., is also commonly used to designate inverse trigonometric functions.

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2. Inverse sine. Remember that angles are expressed in radians. We have chosen x to be in $[-\pi/2,\pi/2]$. Since $x = \sin^{-1}y$ means $\sin x = y$, note that y must be in $[-1,1]$.

3. Trigonometric functions of inverses. Be sure you understand Example 2(b). By drawing an arbitrary right triangle and labelling it appropriately by using the definitions of the trigonometric functions and Pythagoras' theorem, one can easily find the other trigonometric functions.

4. Range for which inverses exist. The angles for which the trigonometric functions have inverses include those in the first quadrant of the xy-plane and the adjacent (second or fourth) quadrant in which the function is negative. If there are two choices, then choose the quadrant which maintains continuity.

5. Derivatives. You may desire to learn to derive the derivatives of the inverses by using the inverse function rule, but it can be time consuming. If memorizing is your forte, you only need to know the derivatives of $\sin^{-1}y$, $\tan^{-1}y$, and $\sec^{-1}y$. The derivatives of the inverses of the cofunctions are the same except for a sign change.

6. Derivative of $\sec^{-1}x$, $\csc^{-1}x$. Notice that the donominator is a square root which is positive. Thus, $\sqrt{y^2(y^2 - 1)}$ simplifies to $|y|\sqrt{y^2 - 1}$. The absolute value sign is a necessity. See Example 8.

7. Antiderivatives. In formula (4) on p. 287, we are given $\int[dx/(1 + x^2)] = \tan^{-1}x + C$. Another valid formula is $\int[dx/(1 + x^2)] = -\cot^{-1}x + C'$. These formulas are equivalent since the arbitrary constants, C and C', may differ.

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SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Since $\sin(0.615) \approx 1/\sqrt{3}$, $\sin^{-1}(1/\sqrt{3}) \approx 0.615$. Remember that the domain of $\sin^{-1}x$ is $[-\pi/2,\pi/2]$.

5. Since $\cos(0) = 1$, $\cos^{-1}(1) = 0$. Remember that the domain of $\cos^{-1}x$ is $[0,\pi]$.

9. $\sec^{-1}(2/\sqrt{3})$ is the same as $\cos^{-1}(\sqrt{3}/2)$, so since $\cos(\pi/6) = \sqrt{3}/2$, $\sec^{-1}(2/\sqrt{3}) = \pi/6$. The domain of $\sec^{-1}x$ is $[0,\pi/2)$ and $(\pi/2,\pi]$.

13. Use the product rule along with the fact that $(d/dx)\sin^{-1}x = 1/\sqrt{1 - x^2}$. Thus, the derivative is $2x \sin^{-1}x + x^2/\sqrt{1 - x^2}$.

17. Combine the chain rule and the quotient rule along with the fact that $(d/dx) \tan^{-1}x = 1/(1 + x^2)$. Thus, the derivative is $[1 + ((2x^5 + x)/(1 - x^2))^2]^{-1} \cdot (d/dx)[(2x^5 + x)/(1 - x^2)] = [(1 - x^2)^2/((1 - x^2)^2 + (2x^5 + x)^2)] \cdot [(1 - x^2)(10x^4 + 1) - (2x^5 + x)(-2x)]/(1 - x^2)^2 = [(1 - x^2)(10x^4 + 1) + 2x^2(2x^4 + 1)]/[(1 - x^2)^2 + x^2(2x^4 + 1)^2]$.

21. Use the quotient rule along with the fact that $(d/dx) \sin^{-1}x = 1/\sqrt{1 - x^2}$. Thus, the derivative is $[(3/\sqrt{1 - 9x^2})(x^2 + 2) - (\sin^{-1}3x) \times (2x)]/(x^2 + 2)^2 = [3(x^2 + 2) - (2x)(\sin^{-1}3x)\sqrt{1 - 9x^2}]/(x^2 + 2)^2\sqrt{1 - 9x^2}$.

25. Apply the quotient rule to get $(d/dx)[(\cos^{-1}x)/(1 - \sin^{-1}x)] = [(-1/\sqrt{1 - x^2})(1 - \sin^{-1}x) - (-1)(1/\sqrt{1 - x^2})(\cos^{-1}x)]/(1 - \sin^{-1}x)^2 = (\sin^{-1}x + \cos^{-1}x - 1)/(1 - \sin^{-1}x)^2\sqrt{1 - x^2}$.

29. Since $(d/dx) \tan^{-1}x = (1 + x^2)^{-1}$, we get $\int [3/(1 + x^2) + x]\,dx = 3 \tan^{-1}x + x^2/2 + C$.

33. $\int [3/(1 + 4x^2)]\,dx = \int [3/(1 + (2x)^2)]\,dx$, so we expect an antiderivative of the form $a[\tan^{-1}(2x)]$. Differentiation implies that $a = 3/2$, so the answer is $3 \tan^{-1}(2x)/2 + C$.

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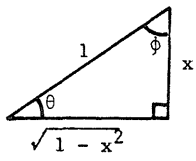
37. 

As shown in the diagram, $\sin^{-1}x = \theta$; therefore, $\tan(\sin^{-1}x) = \tan\theta = x/\sqrt{1 - x^2}$.

41. Since $(d/d\theta)\cos^{-1}\theta = -1/\sqrt{1 - \theta^2} = -(1 - \theta^2)^{-1/2}$, we have $(d^2/d\theta^2)\cos^{-1}\theta = -(-1/2)(1 - \theta^2)^{-3/2}(-2\theta) = -\theta/(1 - \theta^2)^{3/2}$.

45. The rate of change is $(d/ds)\cos^{-1}(8s^2 + 2) = [-1/\sqrt{1 - (8s^2 + 2)^2}](16s)$. At $s = 0$, the derivative is 0 .

49. Use the inverse function rule, $(d/dy)[f^{-1}(y)] = 1/[(d/dx)f(x)]$. Thus, $(d/dy)\cot^{-1}y = 1/(d/dx)\cot x = -1/\csc^2 x = -1/(1 + \cot^2 x) = -1/(1 + y^2)$ because $y = \cot x$.

53. (a) The best method is to simplify and then use implicit differentiation. The equation simplifies to $\sin(x + y) = xy$, which implies $[\cos(x + y)][1 + (dy/dx)] = y + x(dy/dx)$. Therefore, $[\cos(x + y) - y]/[x - \cos(x + y)] = dy/dx$. (Note: $\cos(x + y) = \sqrt{1 - \sin^2(x + y)} = \sqrt{1 - x^2y^2}$, so $dy/dx = (\sqrt{1 - x^2y^2} - y)/(x - \sqrt{1 - x^2y^2})$.)

(b) The chain rule gives us $dy/dt = (dy/dx)(dx/dt)$, where $dx/dt = [(1)(1 - t^2) - (t)(-2t)]/(1 - t^2)^2 = (1 + t^2)/(1 - t^2)^2$. Therefore, $dy/dt = [(\cos(x + y) - y)(1 + t^2)]/[(x - \cos(x + y))(1 - t^2)^2]$.

(c) According to the inverse function rule, $dx/dy = 1/(dy/dx)$. From the chain rule, we get $dx/dt = (dx/dy)(dy/dt)$ where $dy/dt = 1/\sqrt{1 - t^2}$. Therefore, $dx/dt = [x - \cos(x + y)]/[\cos(x + y) - y]\sqrt{1 - t^2}$.

(d) $dx/dt = 3t^2 + 2$. By the method of part (b), we have $dy/dt = [\cos(x + y) - y](3t + 2)/[x - \cos(x + y)]$.

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57. Begin by simplifying the first term, so $\int [(x^2 + 1)/x^2 + 1/\sqrt{1 - x^2} + \cos x]\,dx = \int [1 + x^{-2} + 1/\sqrt{1 - x^2} + \cos x]\,dx = x - 1/x + \sin^{-1}x + \sin x + C$. Evaluating at $\sqrt{2}/2$ and $1/2$, we get $\sqrt{2}/2 - 2/\sqrt{2} + \sin^{-1}(\sqrt{2}/2) + \sin(\sqrt{2}/2) - 1/2 + 2 - \sin^{-1}(1/2) - \sin(1/2) = [\sqrt{2} - 2\sqrt{2} - 1 + 4]/2 + \pi/4 - \pi/6 + \sin(\sqrt{2}/2) - \sin(1/2) = (3 - \sqrt{2})/2 + \pi/12 + \sin(\sqrt{2}/2) - \sin(1/2)$.

SECTION QUIZ

1. Explain why $\sin(\sin^{-1}x) = x$ for all x in the domain of $\sin^{-1}x$, but $\sin^{-1}(\sin x) = x$ only for some x in the domain of $\sin x$.

2. Simplify:

   (a) $\tan(\cos^{-1}x)$

   (b) $\sin 2\theta$ if $\theta = \tan^{-1}(x/2)$

   (c) $\tan^{-1}(-\cos(-\pi))$

3. Why doesn't $\cos(\sin^{-1}(2 + x^2))$ exist?

4. How would you define $\tan^{-1}(\pi/2)$?

5. Your roommate, who has been frantically studying about inverses, was doing the dinner dishes when he imagined seeing little green pea-like beings entering through the kitchen sink. He told you that they introduced themselves, "We're from the Innerverse, the tiny land of inverse trigonometric functions." They had just travelled along the path described by $\sin^{-1}(2x - 2)$.

   (a) The pea-like beings wanted to know the steepness of the path at x_0 for their return trip. Help your roommate get rid of his imaginary friends by providing the correct answer.

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5. (b) Find the width of the path, i.e., determine the length of the domain of $\sin^{-1}(2x - 2)$.

   (c) Your roommate also mentioned that they might return to find out the sine, cosine, and tangent of $\sin^{-1}(2x - 2)$. Compute these quantities for him to help save his sanity.

ANSWERS TO PREREQUISITE QUIZ

1. (a) $\cos x$

   (b) $\sec^2 y$

   (c) $1/11$

2. $(-\infty,0)$ and $(0,\infty)$

3. 1

ANSWERS TO SECTION QUIZ

1. The domain of $\sin^{-1}x$ is only $[-1,1]$, so $\sin^{-1}(\sin x) = x$ only if x is in $[-\pi/2,\pi/2]$.

2. (a) $\sqrt{1 - x^2}/x$

   (b) $4x/(4 + x^2)$

   (c) $\pi/4$

3. $2 + x^2 > 1$ and the domain of $\sin^{-1}x$ is $[-1,1]$.

4. It is θ such that $\tan\theta = \pi/2$, i.e., $\theta \approx 1.00$.

5. (a) $2/\sqrt{1 - (2x_0 - 2)^2}$

   (b) $[1/2,3/2]$

   (c) $\sin(\sin^{-1}(2x - 2)) = 2x - 2$; $\cos(\sin^{-1}(2x - 2)) = \sqrt{1 - (2x - 2)^2}$; $\tan(\sin^{-1}(2x - 2)) = (2x - 2)/\sqrt{1 - (2x - 2)^2}$.

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5.5 Graphing and Word Problems

PREREQUISITES

1. Recall how to solve minimum-maximum word problems (Section 3.5).
2. Recall how to use derivatives to aid in graphing (Section 3.4).
3. Recall how to differentiate trigonometric functions (Section 5.2).
4. Recall how to use the chain rule for differentiation (Section 2.2).
5. Recall the concept of periodicity (Section 5.1).

PREREQUISITE QUIZ

1. Maximize the area of a rectangle whose perimeter is 8 cm.
2. Differentiate:
   (a) $\sin^2 x$
   (b) $\sin x/\sqrt{\cos x}$
   (c) $\cot x$
3. If $f(x) = \sin(4\pi x + 2)$, what is the minimum period?
4. Explain how the first derivative can be used as an aid in graphing.
5. What information is given by the sign of the second derivative?

GOALS

1. Be able to solve word problems involving angles.
2. Be able to graph functions involving trigonometric functions.

STUDY HINTS

1. Typical word problem. Example 1 should be studied carefully. The word "revolution" should immediately translate into "2π radians" in your head. Also, since angles are involved, try to relate given and unknown quantities in terms of trigonometric functions.

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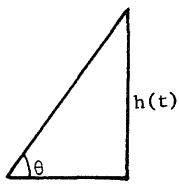
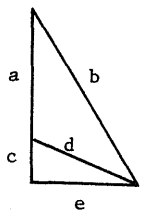
2. Review and practice. Many of these problems rely on the chain rule to relate distances, angles, and time. If necessary, review the chain rule and problem solving techniques from Section 3.5. As always, practice is important.

3. Graphing. The usual techniques of graphing carry over to trigonometric functions. One new aspect is periodicity, which simplifies your work tremendously. Note any symmetry or periodicity first. Note that if a function is a sum of trigonometric functions, the period is the largest of the periods of the terms.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. 

From the diagram, note that $\tan\theta = h(t)/200 = (1000 - 40t - 16t^2)/200$. Consider θ as a function of t and differentiate both sides with respect to t. $\sec^2\theta\ (d\theta/dt) = -(40/200) - (16/200)\,2t$. At $t = 4$, $\tan\theta = [1000 - 40(4) - 16(4)]/200 = 73/25$; hence $\sec^2\theta|_{t=4} = 1 + (73/25)^2 = 9.5264$ because $\sec^2\theta = \tan^2\theta + 1$. Plug in $\sec^2\theta|_{t=4}$ and $t = 4$, and solve for $d\theta/dt$. We get $d\theta/dt|_{t=4} = -0.088$ rad/sec. The negative sign indicates that the angle is decreasing.

5. 

We know that $a + c = 10$ and $a + b = 20$. Differentiating both sides with respect to time, we get $a' + c' = 0$ and $a' + b' = 0$. Therefore, $-a' = b' = c'$. By the Pythagorean theorem, we have $(a + c)^2 + e^2 = b^2$ and $c^2 + e^2 = d^2$. Differentiating $(a + c)^2 + e^2 = b^2$ gives $2(a + c)(a' + c') + 2ee' = 2bb'$ or $2e/b = b'$. Differentiation of $c^2 + e^2 = d^2$ gives

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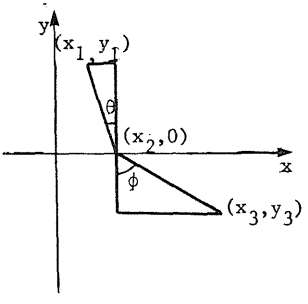
5. (continued)

$2cc' + 2ee' = 2dd'$ or $(cc' + ee')/d = d'$, where $c' = b' = 2e/b$ and e' is given as 2. After 3 seconds, $e = 6$; $b = \sqrt{(10)^2 + (6)^2} = \sqrt{136} \approx 11.66$; $a = 20 - \sqrt{136} \approx 8.34$; $c = 10 - (20 - \sqrt{136}) \approx 1.66$; $d = \sqrt{c^2 + (6)^2} \approx 6.23$; $c' = 1.03$. Therefore, $d' \approx 2.20$ meters/second.

9. We will maximize the square of the distance; $d^2 = (3 \sin 3t - 3 \cos 2t)^2 + (3 \cos 3t - 3 \sin 2t)^2$. Both particles are moving in the same circle of radius 3, but at different speeds; therefore, the maximum distance should be no more than 6. $(d^2)' = 2(3 \sin 3t - 3 \cos 2t)(9 \cos 3t + 6 \sin 2t) + 2(3 \cos 3t - 3 \sin 2t)(-9 \sin 3t - 6 \cos 2t) = 2[-45 \cos 2t \times \cos 3t + 45 \sin 2t \sin 3t] = -90 \cos 5t$. Thus, the critical points are at $(2n + 1)\pi/10$, where n is an integer. At $\pi/10$, $d = 0$, but at $3\pi/10$, $d = 6$, so the maximum distance between A and B is 6. (Notice that if we maximize the distance itself, the derivative is more complicative; however, the critical points are found by solving essentially the same equation.)

13. 

The total time is $T = \sqrt{(x_2 - x_1)^2 + y_1^2} + \sqrt{(x_3 - x_2)^2 + y_3^2}/3$. To minimize the time, we find x_2 such that $dT/dx_2 = 0$. $T' = (1/2) \times [(x_2 - x_1)^2 + y_1^2]^{-1/2} 2(x_2 - x_1) + (1/3)(1/2) \times [(x_3 - x_2)^2 + y_3^2]^{-1/2} 2(x_3 - x_2)(-1)$. Thus, we must solve $(x_2 - x_1)/\sqrt{(x_2 - x_1)^2 + y_1^2} = (1/3)(x_3 - x_2)/\sqrt{(x_3 - x_2)^2 + y_3^2}$. As in Example 2, the left-hand side is $\sin \theta$ and the right-hand side is $(1/3) \sin \phi$. Therefore, we have $3 \sin \theta = \sin \phi$.

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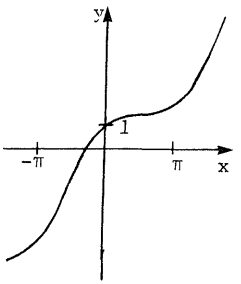
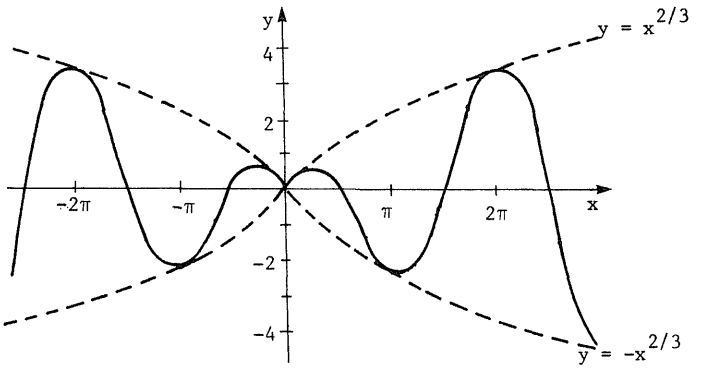
17. $f'(x) = \sin x + x \cos x - 2 \sin x = x \cos x - \sin x$; $f''(x) = \cos x - x \sin x - \cos x = -x \sin x$. $f''(x) = 0$ at $x = \pm n\pi$, n = a non-negative integer. When x is positive, $f''(x) < 0$ for $(0,\pi)$, $(2\pi,3\pi)$, ... , $(2n\pi,(2n + 1)\pi)$. When x is negative, $f''(x) < 0$ for $(-\pi,0)$, $(-3\pi,-2\pi)$, ... , $(-(2n + 1)\pi,-2n\pi)$. Thus, $f(x)$ is concave downward in $(2n\pi,(2n + 1)\pi)$ for $x > 0$ and in $(-(2n + 1)\pi,-2n\pi)$ for $x < 0$. $f(x)$ is concave upward everywhere else.

21. 

$f'(x) = 1 - \sin x$, so the critical points occur at $x = \pi/2 + 2n\pi$, where n is an integer. $f''(x) = -\cos x$, so possible inflection points occur at $x = \pi/2 + n\pi$. Note that the graph oscillates around the line $y = x$.

25. $x^{2/3}\cos x$ is symmetric about the y-axis. Its graph oscillates between $\pm x^{2/3}$. Consider $f(x) = x^{2/3}$: $f'(x) = 2/3x^{1/3}$ and $f''(x) = -2/9x^{4/3}$. There is a cusp at $x = 0$ and the graph of $x^{2/3}$ is concave down. The graph of $x^{2/3}\cos x$ looks like $\cos x$ except for the scaling factor.



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29. Recall that the equation of a line is $y = y_1 + m(x - x_1)$, where m is the slope. $f'(x) = -2/\sqrt{1 - 4x^2} - 1/\sqrt{1 - x^2}$, which is -3 at $x = 0$. Therefore, the slope of the tangent line is -3 and the slope of the normal line is the negative reciprocal, $1/3$. Thus, the tangent line is $y = -3x + \pi$ and the normal line is $y = x/3 + \pi$.

33. $f'(x) = (x \cos x - \sin x)/x^2$. The critical points occur where $x \cos x = \sin x$ or $x = \tan x$, which has infinitely many solutions. As $x \to \infty$, $\tan x$ also approaches ∞ . This also occurs near $\pi/2 + n\pi$, where n is an integer. Thus, the local maxima and minima are about π units apart as $x \to \infty$.

SECTION QUIZ

1. Sketch $y = \sqrt{x} \cos x + \cos \pi$.

2. You and your partner have just finished cutting down a 100 m. redwood tree. As you yell "Timberr...", your partner realizes that the tree is about to fall on top of him. Without thinking, he makes a mad dash in the path of the tree's shadow. If the tree is falling at a rate of 9°/sec., how fast is it falling vertically after falling an angle of $\pi/4$? How fast is the shadow increasing assuming the sun is directly overhead? (Your friend is a world class sprinter and barely escaped being crushed.)

3. Those silly sisters, Sandra and Sheila, are at it again. On a pleasant Spring day, they decided to go ballooning. As Sandra unties the rope anchoring the balloon, a sudden gust of wind whisks the balloon away, parallel to the ground with Sheila in it at a speed of 1 m./sec. In the meantime, Sandra, who has been hanging on for dear life, falls to

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3. (continued)

the ground, 25 meters below the balloon. She gets up and runs after Sheila at a speed of 0.25 m/sec. After one minute, Sandra tires and yells, "Bye, Sheila. I'll send the Air Force to rescue you." How fast is the angle of Sandra's eyes changing one minute after she quit running if she continues to keep an eye on the balloon?

ANSWERS TO PREREQUISITE QUIZ

1. $4\ cm^2$ (2 cm. × 2 cm.)

2. (a) $2 \sin x \cos x$

   (b) $\sqrt{\cos x} + \sin^2 x/2 \cos^{3/2} x$

   (c) $-\csc^2 x$

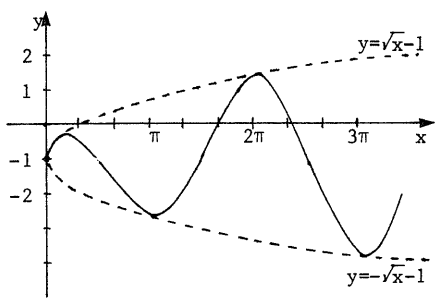
3. 1/4

4. If $f'(x) < 0$, the function is decreasing. If $f'(x) > 0$, the function is increasing.

5. If $f''(x) < 0$, the function is concave downward. If $f''(x) > 0$, the function is concave upward.

ANSWERS TO SECTION QUIZ

1.



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2. $5\pi/\sqrt{2}$ m./sec. ; $5\pi/\sqrt{2}$ m./sec.

3. -1/2650 rad./sec.

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5.6 Graphing in Polar Coordinates

PREREQUISITES

1. Recall the first derivative test (Section 3.2).
2. Recall how to calculate with polar coordinates (Section 5.1).

PREREQUISITE QUIZ

1. Convert the cartesian coordinates $(4,2)$ to polar coordinates.
2. Convert the polar coordinates $(-3,-3\pi/4)$ to cartesian coordinates.
3. What do you know about a function f if $f'(x_0) > 0$?

GOALS

1. Be able to graph a function given in polar coordinates on the xy-plane.
2. Be able to find the slope of a function given in polar coordinates.

STUDY HINTS

1. Symmetry. As with graphing of functions in the xy-plane, symmetry can be used to eliminate much work. Example 1 shows how to determine symmetry in the x- and y-axes.
2. Rotational symmetry. Note that if $f(\theta + \phi) = f(\theta)$ for all θ, then the graph is unchanged after a rotation of ϕ radians.
3. Cartesian vs. polar coordinates graphing. The graph of $r = f(\theta)$ can be graphed in two ways. Graphing in the $r\theta$-plane was discussed in Section 5.1 and graphing in the xy-plane is discussed in this section. Be careful to read what is asked for.
4. Rose petals. Examples 1 and 2 illustrate a general fact about $r=\sin n\theta$ and $r=\cos n\theta$. If n is odd, the graph is an n-petaled rose. If n is even, the graph is a $2n$-petaled rose.

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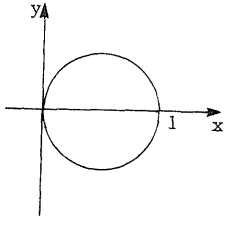
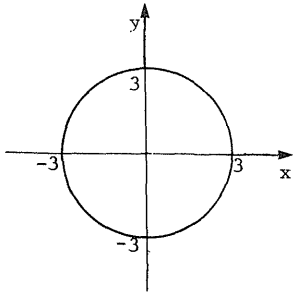
5. Tangents in polar coordinates. Rather than memorizing the formula $dy/dx = [(\tan\theta)dr/d\theta + r]/[(dr/d\theta) - r\tan\theta]$, you may find it easier to derive the formula. Just remember that $dy/dx = (dy/d\theta)/(dx/d\theta)$ and use the chain rule to differentiate $x = r\cos\theta$ and $y = r\sin\theta$.

6. Interpretation of maxima, minima. Sometimes, a minimum represents a maximal distance from the origin because it is usually a negative number and distances are absolute values. A minimum represents an actual minimal distance if it is positive. What are the corresponding statements for maxima?

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. 

In cartesian coordinates, we have $r = \sqrt{x^2 + y^2} = x/\sqrt{x^2 + y^2} = \cos\theta$ or $x^2 + y^2 = x$. Rearrangement yields $(x^2 - x + 1/4) + y^2 = (x - 1/2)^2 + y^2 = 1/4$. So the graph is a circle centered at $(1/2,0)$ with radius $1/2$.

5. 

$r = 3$ implies $r^2 = 9$ or $x^2 + y^2 = 9$. This is a circle of radius 3 centered at the origin.

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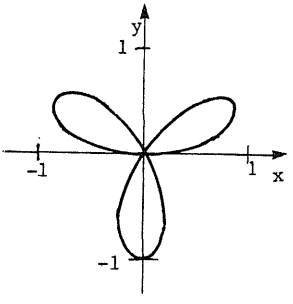
9. 

$r = \sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta \cos \theta + \cos 2\theta \sin \theta = (2 \sin \theta \cos \theta)\cos \theta + (\cos^2\theta - \sin^2\theta) \sin \theta = 3 \sin \theta \cos^2\theta - \sin^3\theta$. Multiply both sides by r^3 to get $r^4 = 3r^3 \sin \theta \cos^2\theta - r^3 \sin^3\theta$. Substituting $r^2 = x^2 + y^2$, $x = r \cos \theta$, and $y = r \sin \theta$, we get $(x^2 + y^2)^2 = 3x^2y - y^3$. The graph of $\sin 3\theta$ is exactly like $\cos 3\theta$ except that $\sin 3\theta$ is shifted $(\pi/2)/3 = \pi/6$ radians to the right. The graph of $r = \sin 3\theta$ is obtained by rotating $r = \cos 3\theta$ (Example 2) by $\pi/6$.

13. $f(\pi/2 - \theta) = f(\theta)$ implies that $f(\theta)$ is symmetric with respect to the diagonal line $\theta = \pi/4$. Hence, the graph of $r = f(\theta)$ is symmetric with respect to the line $\theta = \pi/4$.

17. Use the relations $x = r \cos \theta$ and $y = r \sin \theta$ to get $1 = x^2 + xy + y^2 = r^2 \cos^2\theta + (r \cos \theta)(r \sin \theta) + r^2 \sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) + r^2 \cos \theta \sin \theta = r^2(1 + \cos \theta \sin \theta)$.

21. Use the relations $x = r \cos \theta$ and $y = r \sin \theta$ to get $y = r \sin \theta = r \cos \theta + 1 = x + 1$. This simplifies to $r(\sin \theta - \cos \theta) = 1$.

25. The slope of the tangent line to the graph $r = f(\theta)$ at (r,θ) is $[(\tan \theta)(dr/d\theta) + r]/[dr/d\theta - r \tan \theta]$. Here, $dr/d\theta = 10 \cos 5\theta$. At $\theta = \pi/2$, $r = 2 \sin 5\theta = 2$ and $dr/d\theta = 0$, so the slope is $[(\tan \theta)(0) + 2]/[0 - 2 \tan \theta]$. Since $\tan \theta \to \infty$, we need to find the limit. Divide by $\tan \theta/\tan \theta$ to get $(0 + 2/\tan \theta)/(-2)$, which has a limit of 0 .

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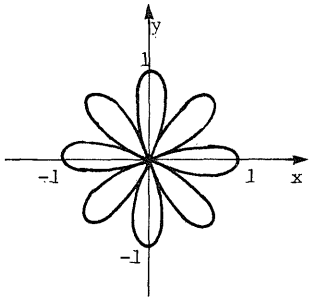
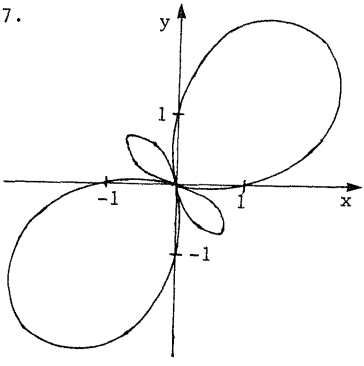
29. The slope of the tangent line to the graph $r = f(\theta)$ at (r,θ) is $[(\tan\theta)(dr/d\theta) + r]/[dr/d\theta - r\tan\theta]$. Here, $dr/d\theta = 3\cos\theta - 2\theta\sin(\theta^2)$. At $\theta = 0$, $r = 1$, $\tan\theta = 0$, and $dr/d\theta = 3$. Thus, the slope is $[0(3) + 1]/[3 - (1)(0)] = 1/3$.

33. 

Note that $dr/d\theta = -4\sin 4\theta$, which is 0 if $\theta = n\pi/4$ where n is an integer. $d^2r/d\theta^2 = -16\cos 4\theta$ which is negative for even n and positive for odd n . Hence $r = 1$ at $\theta = n\pi/2$, which are the local maxima and $r = -1$ at $\theta = (2n + 1)\pi/4$, which are the local minima.

37. 

Note that $dr/d\theta = 4\cos 2\theta$, which is 0 if $2\theta = (2n + 1)\pi/2$, i.e., $\theta = (2n + 1)\pi/4$ for all integers n . Also, $d^2r/d\theta^2 = -8\sin 2\theta$. For n even, $\theta = (2n + 1)\pi/4$ is a local maximum point with $r = 3$; for n odd, $\theta = (2n + 1)\pi/4$ is a local minimum point with $r = -1$.

SECTION QUIZ

1. How is the graph of $\cos 2(\theta + 7\pi/8)$ related to the graph of $\cos 2\theta$?
2. If $r = \csc\theta$, what is the equation of the tangent line at $\theta = \pi/4$?
3. Sketch the graph of $\sin^{-1}r = 2\theta$. Are there any restrictions on r ?
4. In a hospital waiting room, an expectant father was seen pacing and making a track in the carpet. Another gentleman, waiting for his fourth child, notices that the track can be described by $r = \sin(\theta + \pi/6)$.

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4. (a) Sketch the curve.

   (b) When the nurse enters and says, "Congratulations, they're triplets," our new father runs off to buy cigars. His path is a tangent from the point $(r,\theta) = (1/2,4\pi)$. What equation describes the path?

ANSWERS TO PREREQUISITE QUIZ

1. $(\sqrt{20},\tan^{-1}0.5)$

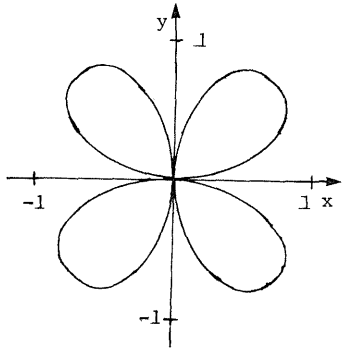
2. $(3/\sqrt{2},3/\sqrt{2})$

3. f is increasing at x_0 .

ANSWERS TO SECTION QUIZ

1. The graph of $\cos 2(\theta + 7\pi/8)$ is the same as the graph of $\cos 2\theta$ after a clockwise rotation of $7\pi/8$ radians.

2. $y = 1$

3. 

The restriction is $-1 \leqslant r \leqslant 1$. The graph is that of $r = \sin 2\theta$.

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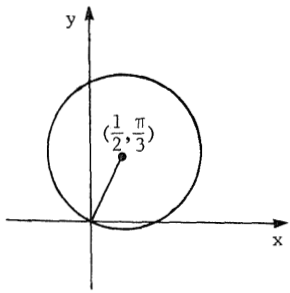
4. (a)



(b) $y = (x - 1/2)/\sqrt{3}$

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5.S Supplement to Chapter 5: Length of Days

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Using the first-order approximation, $\Delta S \approx dS/dT$ with dS/dT given by formula (4) and $\ell = 33.57^\circ$, $\alpha = 0.41$ radian and $T = 22$ (June 21 is day 1). Note that $2\pi T/365$ is in radians. We get $dS/dT|_{T=22} = -0.0083$ hours ≈ -30 seconds. Hence on July 13, the sun sets at about 8:05:30. On July 14, we can use the same formula for dS/dT with $T = 23$. We get $dS/dT|_{T=23} = -0.0086$ hour ≈ -31 seconds. Hence on July 14, the sunsets at about 8:05:00.

5. We use the formula: $\sin A = \cos\ell\sqrt{1 - \sin^2\alpha\cos^2(2\pi T/365)}\cos(2\pi t/24) + \sin\ell\sin\alpha\cos(2\pi T/365)$. At sunset, $A = 0$ and $t = 6.5$, so we get $\cos\ell\sqrt{1 - \sin^2\alpha\cos^2(2\pi T/365)}\cos(13\pi/24) - \sin\ell\sin\alpha\cos(2\pi T/365)$, which implies $\sqrt{1 - \sin^2\alpha\cos^2(2\pi T/365)}/\cos(2\pi T/365) = -\tan\ell\sin\alpha/\cos(13\pi/24)$. Squaring gives us $1/\cos^2(2\pi T/365) - \sin^2\alpha = [\tan\ell\sin\alpha/\cos(13\pi/24)]^2$. Adding $\sin^2\alpha$ to both sides, taking the square root, the reciprocal, and then arccos gives us $2\pi T/365 = \cos^{-1}[\tan^2\ell\sin^2\alpha/\cos(13\pi/24) + \sin^2\alpha]^{-1/2}$. Multiplying through by $365/2\pi$ gives us $T = (365/2\pi)\cos^{-1}\sqrt{\cos^2(13\pi/24)/[\sin^2\alpha(\tan^2\ell + \cos^2(13\pi/24))]} = (365/2\pi)\cos^{-1}\sqrt{0.107/(\tan^2\ell + 0.017)}$. The domain requires the expression under the radical to be in $[0,1]$, so we solve $0 \leq \cos^2(13\pi/24)/[\sin^2\alpha(\tan^2\ell + \cos^2(13\pi/24))] \leq 1$. Divide by $\cos^2(13\pi/24)/\sin^2\alpha$ and take the reciprocal to get $\infty > \tan^2\ell + \cos^2(13\pi/24) \geq \cos^2(13\pi/24)/\sin^2\alpha$. Subtract $\cos^2(13\pi/24)$, take the square root and then the arctangent. Thus, the domain is $[-\pi/2,-0.29]$ and $[0.29,\pi/2]$. We plot the following points:

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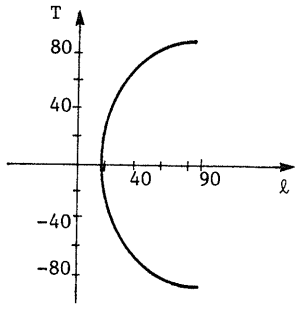
5. (continued)

$\ell(^\circ)$	16.71	17	18	20	25	30	40	50
$\pm T$ (days)	0.0	10.2	21.1	32.6	48.1	57.2	68.3	75.2

$\ell(^\circ)$	60	70	80	90
$\pm T$ (days)	80.2	84.3	87.9	91.3

The other half of the graph is obtained by a reflection across the T-axis.



9. (a) We use the formula for the length of days. For this problem, we have $\ell = 0$ because the location is at the equator; $T = 0$ by definition; and dA/dt is the quantity of interest. Therefore, the equation reduces to $\sin A = \sqrt{1 - \sin^2\alpha}[\cos(2\pi t/24)]$. Differentiating both sides with respect to t gives $\cos A(dA/dt) = \sqrt{1 - \sin^2\alpha}[-\sin(2\pi t/24)](2\pi/24)$. The formula tells us that all days at the equator are 12 hours long, so $t = 18$ at sunrise. Also, we know that $A = 0$ at sunrise, so $(dA/dt) = (\pi/12) \times \sqrt{1 - \sin^2\alpha} = 0.24$ radians/hour or 13.76°/hour .

(b) The linear approximation tells us that $\Delta A = (dA/dt)\Delta t$, so $\Delta t = \Delta A/(dA/dt) = 5^\circ/(13.76^\circ/\text{hr.}) = 0.36$ hours = 21 minutes, 49 seconds.

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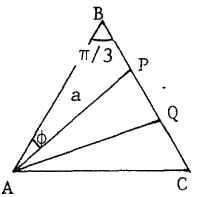
5.R Review Exercises for Chapter 5

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Since π radian $= 180^\circ$, we get $66^\circ \times (\pi/180) = 1.152$ radians.

5. Use $x = r\cos\theta$ and $y = r\sin\theta$ to get $y = r\sin\theta = r^2\cos^2\theta = x^2$ or $\tan\theta\sec\theta = r$.

9. By the definition of sine, we have $\sin 8^\circ = a/10$, so $a = 10\sin 8^\circ \approx 1.392$.

13. 

By the law of cosines, $a^2 = 1^2 + 3^2 - 2(1)(3)\cos(\pi/3) = 7$ so $a = \sqrt{7}$. Applying the law of cosines again, $1^2 = 3^2 + (\sqrt{7})^2 - 2(3)(\sqrt{7})\cos\phi$ implies $\cos\phi = 15/6\sqrt{7}$, so $\phi \approx 0.333$. By symmetry, the lower angle is also 0.333 and since all of the angles of an equilateral triangle are $\pi/3$, the remaining angle is 0.380 .

17. Since $(d/dx)\sin x = \cos x$, we get $dy/dx = 1 + \sin 3x + 3x\cos 3x$.

21. Using the fact that $(d/dy)\tan y = \sec^2 y$, we get $h'(y) = 3y^2 + 2\tan(y^3) + 2y\sec^2(y^3)\cdot(3y^2) = 3y^2 + 2\tan(y^3) + 6y^3\sec^2(y^3)$.

25. Using the facts that $(d/dx)\sec^{-1}x = 1/x\sqrt{x^2 - 1}$ and $(d/dx)\sin x = \cos x$, we get $f'(x) = \{1/[(x + \sin x)^2\sqrt{(x + \sin x)^4 - 1}]\}\cdot 2(x + \sin x)\times(1 + \cos x) = 2(1 + \cos x)/(x + \sin x)\sqrt{(x + \sin x)^4 - 1}$.

29. Since $(d/dx)\sin^{-1}x = 1/\sqrt{1 - x^2}$, we get $(1/\sqrt{1 - x})(d/dx)\sqrt{x} = 1/2\sqrt{x}\sqrt{1 - x} = 1/2\sqrt{x - x^2}$.

33. Using the fact that $(d/dx)\sin^{-1}x = 1/\sqrt{1 - x^2}$ and $(d/dx)\cos x = -\sin x$, we get $(d/dx)\sin^{-1}(\sqrt{x} + \cos 3x) = [1/\sqrt{1 - (\sqrt{x} - \cos 3x)^2}]\cdot(1/2\sqrt{x} - 3\sin 3x)$.

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37. By the product rule, $dh/dx = \sin^{-1}(x + 1) + x/\sqrt{1 - (x + 1)^2}$. By the chain rule, $dh/dy = (dh/dx)(dx/dy) = \sin^{-1}(x + 1) + x/\sqrt{1 - (x + 1)^2}] \times (1 - 3y^2) = \sin^{-1}(y - y^3 + 1) + (y - y^3)/\sqrt{1 - (y - y^3 + 1)^2}](1 - 3y^2)$.

41. Guess that $\int \sin 3x\, dx = a \cos 3x + C$. Differentiation yields $a = -1/3$; therefore, $\int \sin 3x\, dx = -\cos 3x/3 + C$.

45. Guess that $\int (3x^2 \sin x^3)dx$ has the form $a \cos x^3 + C$. Differentiation shows that $a = -1$, so the integral is $-\cos x^3 + x^2 + C$.

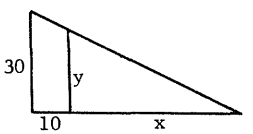
49. Factor out $1/4$, so $(1/4)\int [dy/(1 + (y/2)^2)]$ should have the form $a \tan^{-1}(y/2)$. Differentiation shows that $a = 1/2$; therefore, $\int [1/(4 + y^2)]\, dy = (1/2) \tan^{-1}(y/2) + C$.

53. Factor out $1/2$ to get $\int_{-1}^{1} [1/\sqrt{4 - x^2}]\, dx = (1/2)\int_{-1}^{1} [1/\sqrt{1 - (x/2)^2}]\, dx$. The antiderivative should have the form $a \sin^{-1}(x/2) + C$. Differentiation yields $(a/2)/\sqrt{1 - (x/2)^2}$, so $a = 1$ and the integral is $\sin^{-1}(x/2)\Big|_{-1}^{1} = \pi/6 - (-\pi/6) = \pi/3$.

57. (a) We want to find an interval in which $f(x)$ is strictly increasing or decreasing. $f'(x) = 3x^2 - 3$ implies the critical points are ± 1 ; therefore, the interval containing zero is $[-1,1]$.

(b) $f'(x) < 0$ in $[-1,1]$, so it is decreasing and the domain of g is $[f(b),f(a)] = [5,9]$.

(c) By inspection, $f(0) = 7$, so $g(7) = 0$. Therefore, by the inverse function rule, $g'(7) = 1/[f'(0)] = 1/(3x^2 - 3)\Big|_0 = -1/3$.

61. 

By similar triangles, $30/(10 + x) = y/x$, which implies $30x = y(10 + x)$. Upon differentiating with respect to t , we have $30x' = y'(10 + x) + yx'$, and so $x' = y'(10 + x)/(30 - y)$. After 4 seconds, $y = 8$, $x = 40/11$, and $y' = 2$. Therefore, $x' = 2(10 + 40/11)/(30 - 8) = 2(150/11)/22 = (150/121)$ m./sec.

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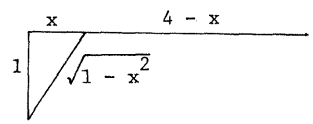
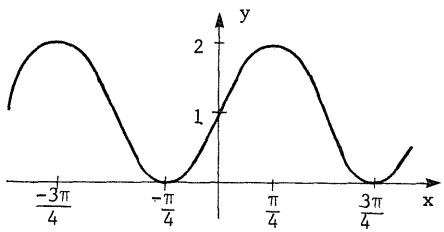
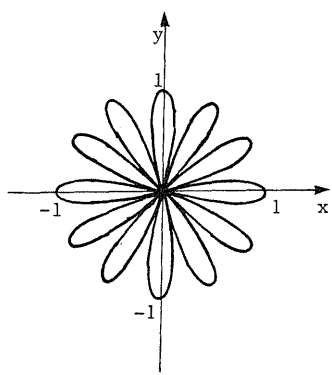
65. 

Time is distance/speed, so we want to minimize $T = \sqrt{1 + x^2}/4 + (4 - x)/16$. $T'(x) = (1/2)(1 + x^2)^{-1/2}(2x)/4 - 1/16 = (4x - \sqrt{1 + x^2})/16\sqrt{1 + x^2}$. Setting $T'(x) = 0$, we get $4x = \sqrt{1 + x^2}$ or $16x^2 = 1 + x^2$; therefore, $x = (1/\sqrt{15})$ km is the point to where she should row.

69. 

$f'(x) = 2[(\cos x)(\cos x) + (\sin x)(-\sin x)] = 2(\cos^2 x - \sin^2 x) = 2 \cos 2x$, so the critical points are $(2n + 1)\pi/4$, where n is an integer. $f''(x) = -4 \sin 2x$, so the inflection points are $n\pi/2$. $f''(x) < 0$ in $[n\pi, (2n + 1)\pi/2]$, so $f(x)$ is concave down in this interval and concave up elsewhere. Since $\sin(x + \pi) = -\sin x$ and $\cos(x + \pi) = -\cos x$, we have $f(x + \pi) = f(x)$. Therefore, the period is π and we only have to graph $0 \leqslant x \leqslant \pi$ and then use the symmetry properties.

73. 

We note that $f(\theta) = f(-\theta)$; $f(\pi - \theta) = \cos(6\pi - 6\theta) = \cos(-6\theta) = \cos 6\theta = f(\theta)$; $f(\pi + \theta) = \cos(6\pi + 6\theta) = \cos(6\theta) = f(\theta)$. Thus, the graph is symmetric in the x-axis, the y-axis, and the origin, so we need only plot the graph in $[0, \pi/2]$ and use reflections. $dr/d\theta = -6 \sin 6\theta$ implies that critical points occur at 0 , $\pi/6$, $\pi/3$, $\pi/2$. The zeros occur at $\pi/12$, $\pi/4$, $5\pi/12$.

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77. The tangent line is given by $y = y_0 + m(x - x_0)$, where $y_0 = r \sin \theta$, $x_0 = r \cos \theta$, and $m = [(\tan \theta)(dr/d\theta) + r]/[dr/d\theta - r \tan \theta]$. Here $dr/d\theta = -4 \sin 4\theta$. At $\theta = \pi/4$, $\tan(\pi/4) = 1$, $r = \cos \pi = -1$, $x_0 = (-1) \cos (\pi/4) = -\sqrt{2}/2$, and $y_0 = (-1) \sin(\pi/4) = -\sqrt{2}/2$. Thus, $m = [(1)(0) + (-1)]/[0 - (-1)(1)] = -1$, and the tangent line is $y = (-\sqrt{2}/2) + (-1)[x - (-\sqrt{2}/2)] = -x - \sqrt{2}$.

81. We use the chain rule and the inverse function rule. Let $y = \sqrt{x}$, then the chain rule tells us that $g'(x) = (dg/dy) \cdot (dy/dx)$. Now according to the inverse function rule, $dg/dy = 1/f'(g(y)) = 1/f'(f^{-1}(y)) = 1/f'(f^{-1}(\sqrt{x}))$, and by ordinary differentiation, $dy/dx = 1/2\sqrt{x}$. Therefore, $g'(x) = 1/2f'(f^{-1}(\sqrt{x}))\sqrt{x}$.

85. If $x \neq 0$, then $f(x) = x^4 \sin(1/x)$ and $f'(x) = 4x^3 \sin(1/x) + x^4(-1/x^2)\cos(1/x) = 4x^3 \sin(1/x) - x^2 \cos(1/x)$. Therefore, the chain rule and the product rule may be applied again to get $f''(x) = 12x^2 \sin(1/x) + 4x^3(-1/x^2) \cos(1/x) - 2x \cos(1/x) - x^2(-1/x^2)(-\sin(1/x)) = 12x^2 \sin(1/x) - 6x \cos(1/x) - \sin(1/x)$. Since the last term has no limit as $x \to 0$, $f''(x)$ has no limit as $x \to 0$.

On the other hand, $f'(0) = \lim_{x \to 0}\{[f(x) - f(0)]/x\} = \lim_{x \to 0}\{[x^4 \sin(1/x)]/x\} = \lim_{x \to 0}[x^3 \sin(1/x)]$. This limit is zero since $|\sin(1/x)| \leq 1$. Also, $f''(0) = \lim_{x \to 0}\{[f'(x) - f'(0)]/x\} = \lim_{x \to 0}\{[4x^3 \sin(1/x) - x^2 \cos(1/x)]/x\} = \lim_{x \to 0}[4x^2 \sin(1/x) - x \cos(1/x)]$. Again, the limit is 0 since $|\sin(1/x)| \leq 1$ and $|\cos(1/x)| \leq 1$. Thus, f is indeed twice differentiable, but f'' is not continuous since $\lim_{x \to 0} f''(x)$ does not exist.

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TEST FOR CHAPTER 5

1. True or false.

   (a) The origin can have any θ in polar coordinates.

   (b) In polar coordinates, $\sin\theta = 1$ and $\sin\theta = -1$ describe the same line.

   (c) $(d/dx)\ \sin^{-1}x = -\cos x/\sin^2 x$.

   (d) $y = x^5$ has an inverse on $(-\infty,\infty)$, but $y = x^4$ does not.

   (e) If $f'(x) < 0$ on $[a,b]$, then the domain of f^{-1} is $[f(a),f(b)]$.

2. Differentiate:

   (a) $(\cos\sqrt{x}\ \sin x)^2$

   (b) $\sec^{-1}(2x)$

   (c) $\tan^{-1}(5x + 2)$

   (d) $f^{-1}(x)$ if $f(x) = x + (1/2)\cos x$

3. Integrate:

   (a) $\int_0^{\pi}\sin t\ dt$

   (b) $\int_0^1[d\theta/(1 + \theta^2)]$

   (c) $\int \csc x\ \sin^2 x\ dx$

4. (a) Sketch the graph of $y = \cos 2(x + 3\pi/8)$.

   (b) Find intervals for which an inverse exists.

   (c) Sketch the inverse on the interval which contains $x = \pi/4$.

5. (a) Sketch the graph of $r = \cos 2(\theta + 3\pi/8)$ in the xy-plane. (Compare this sketch with that of Question 4(a).)

   (b) Convert this equation to cartesian coordinates.

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6. In each case, find $\sin 2\theta$ for the given information.

   (a) $\sin \theta = 5/6$

   (b) $\tan^{-1} x = \theta$

7. A particle follows the path given by the equation $r = 2 \cos \theta + \sin \theta$.

   (a) If it flies off along the tangent line at $\theta = \pi/4$, what is the equation of the tangent line?

   (b) Will the particle hit the x-axis, y-axis, both, or neither, assuming the particle traverses a counterclockwise path? Where will it hit?

8. Let $x = \pi/4 + 0.2$. Approximate $\cos x$, $\sin x$, and $\tan x$. Use $\sqrt{2} \approx 1.4$.

9. Let $x = 2t$ and $y = \cos^{-1} t$.

   (a) What is the domain?

   (b) Sketch the curve in the xy-plane.

10. Steve, the fearless skydiver, was attempting to dive into a red target. 250 m. above the target, he was falling at 10 m./sec. Unfortunately, skydiving Steve's friends placed the target in the middle of a bull pasture. One angry bull stood 50 m. from the target. How fast is the bull's head rotating downward as he keeps an eye on Steve?

ANSWERS TO CHAPTER TEST

1. (a) True

   (b) True

   (c) False; $(d/dx) \sin^{-1} x = 1/\sqrt{1 - x^2}$.

   (d) True

   (e) False; if $f'(x) < 0$, the domain is $[f(b), f(a)]$.

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2. (a) $2(\cos \sqrt{x} \sin x)(-\sin \sqrt{x} \sin x/2\sqrt{x} + \cos \sqrt{x} \cos x)$

   (b) $1/x\sqrt{4x^2 - 1}$

   (c) $5/[(5x + 2)^2 + 1]$

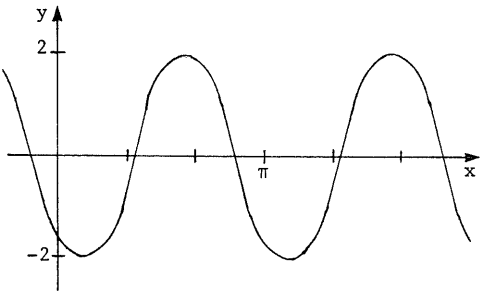
   (d) $1/(1 - (1/2)\sin x)$

3. (a) 2

   (b) $\pi/4$

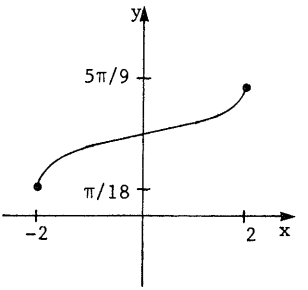
   (c) $-\cos x + C$

4. (a)



   (b) $(n\pi/2 , (n + 1)\pi/2)$, where n is an integer.

   (c)



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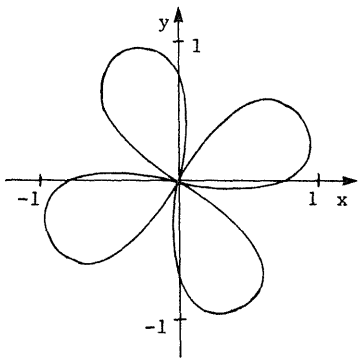
5. (a)



(b) $\sqrt{\Sigma} = (y^2 - 2xy - x^2)/(x^2 + y^2)^{3/2}$

6. (a) $5\sqrt{11}/18$

(b) $2x/(1 + x^2)$

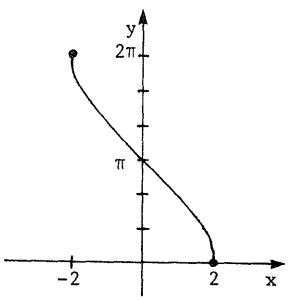
7. (a) $2y + x = 9/2$

(b) y-axis at $(0,9/4)$

8. $\cos(\pi/4 + 0.2) \approx 0.56$; $\sin(\pi/4 + 0.2) \approx 0.84$; $\tan(\pi/4 + 0.2) \approx 1.40$

9. (a) $-1 \leqslant t \leqslant 1$ or $-2 \leqslant x \leqslant 2$

(b)



10. (1/130) rad/sec

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